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旋转变厚度圆盘的样条积分方程法

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一、前 言

对于旋转变厚度圆盘的应力分析，一般可用有限元法或差分法，但相对于边界元来说精度较差，若要提高精度就必须将单元分细，随之而来的是方程阶数的增加和计算时间的增长。为了克服上述缺点，本文提出了一种新的样条积分方程法，直接利用弹性力学二维问题的环基本解建立积分方程并用样条函数进行求解。这样只需划分少量单元，就可获得精度很高的数值解，可大大节约计算机内存和计算时间。

二、平衡方程及二维弹性力学问题的环基本解

1. **平衡方程** 对以角速度 ω 旋转的变厚度 t（t 是座标 r 的函数）圆盘，由环向、径向切出的微元体可导出以位移 u 表示的平衡方程

$$\frac{E}{1-v^2}\left[\frac{d^2u}{dr2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}\right]+\frac{t'}{t}\frac{E}{1-v^2}\left[\frac{du}{dr}+v\frac{u}{r}\right]+\rho_1\omega^2 r=0 \tag{1}$$

2. **二维弹性力学问题的环基本解** u^* u^* 应满足方程

$$\frac{E}{1-v^2}\left[\frac{d^2u^*}{dr2}+\frac{1}{r}\frac{du^*}{dr}-\frac{u^*}{r^2}\right]+\frac{\delta(r-\rho)}{2\pi\rho}=0 \tag{2}$$

相应于方程（2）的齐次方程为

$$d^2u^*/dr^2+du^*/rdr=0 \tag{3}$$

对（3）进行两次积分求得通解，然后利用 $r=0$、$r\to\infty$ 时 u^* 为零的自然条件及 $r=\rho$ 时的位移连续和应力平衡条件，可得到二维弹性力学问题的环基本解

（1）当 $r\geqslant\rho\geqslant 0$ 时，

$$u^*=(\rho/r)(1-v^2)/4\pi E$$

$$\sigma_r^*=(\rho/r^2)(v-1)/4\pi,\qquad \sigma_\theta^*=(\rho/r^2)(1-v)/4\pi \tag{4}$$

（2）当 $0\leqslant r\leqslant\rho$ 时，

$$u^*=(r/\rho)(1-v^2)/4\pi E$$

$$\sigma_r^*=(1+v)/4\pi\rho,\qquad \sigma_\theta^*=(1+v)/4\pi\rho \tag{5}$$

三、旋转变厚度圆盘的样条积分法

1. **积分方程的建立** 设旋转变厚度圆盘的内半径为 b，外半径为 a，假定在区间（b, a）上，厚度 t 连续并且 t' 存在，那么在区间（b, a）上 u 及 u' 连续，u'' 存在，如果 t 不满足连续条件，可将（b, a）划分成若干个子区间分别求解。

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用格林第二公式及弹性力学二维问题的环基本解，将方程（1）写成如下形式的积分方程

$$u(\rho) = 2\pi a(\sigma_{ra}u_{ra}^{*} - \sigma_{ra}^{*}u_{ra}) - 2\pi b(\sigma_{rb}u_{rb}^{*} - \sigma_{rb}^{*}u_{rb}) + \int_b^a \frac{2\pi r t'}{t}\frac{E}{1-v^2}\left[\frac{du}{dr} + v\frac{u}{r}\right]u^{*}dr + \int_b^a 2\pi r^2\rho_1\omega^2 u^{*}dr \tag{6}$$

相应的边界条件为

$$\sigma_{ra} = 0 \quad 或 \quad \left.\frac{du}{dr}\right|_{r=a} = -v\frac{u_{ra}}{a},\ \sigma_{rb} = 0 \quad 或 \quad \left.\frac{du}{dr}\right|_{r=b} = -v\frac{u_{rb}}{a} \tag{7}$$

如果是实心圆盘，只需在方程（6）中令 $b=0$ 即可，并补充 $r=0$ 处条件

$$u|_{r=0} = 0 \tag{8}$$

2. **样条插值** 把区间（0，a）或（b，a）划分成 N 等分，利用三次样条插值可得

$$u(r) = \sum_{j=-1}^{N+1} c_j\Omega_3\frac{r-r_j}{h} \tag{9}$$

式中 c_j 为系数，Ω_3（x）为三次 B 样条函数，而 $h=a/N$ 或（$a-b$）/N，$r_j=aj/N$ 或 $b+$（$a-b$）j/N（$j=-1$，0，1，…，$N+1$）

利用结点位移条件可得系数 c_j 和结点位移关系为

$$\boldsymbol{C} = \boldsymbol{A}^{-1}\boldsymbol{X} \tag{10}$$

$$\boldsymbol{C} = [c_{-1}, c_0, c_1, \cdots, c_{N+1}]^T, \boldsymbol{X} = [u_0', u_0, \cdots, u_N, u_N']^T \tag{11}$$

矩阵 $\boldsymbol{A}$ 为（$N+3$）×（$N+3$）矩阵[5]。

令 $\overline{\Omega}_3=$ [Ω_3（$r-r_{-1}$）/h，Ω_3（$r-r_0$）/h，…，Ω_3（$r-r_{N+1}$）/h]

则 $u=\overline{\Omega}_3\boldsymbol{A}^{-1}\boldsymbol{X}$， $u'=\overline{\Omega}_3'\boldsymbol{A}^{-1}\boldsymbol{X}$ （12）

3. **方程的求解** 将（12）代入（6）并注意到边界条件（7）或（8）可知共有（$N+1$）个未知量，只要令方程（6）中的 $\rho=\rho_j=aj/N$ 或 $b+$（$a-b$）j/N（$j=0$，1，2，…，N）可得到（$N+1$）个联立方程组，求解这组方程组就可求得这（$N+1$）个未知量。求得未知量后可由（12）式并结合弹性力学本构方程可得到任一点处的位移和应力。

四、计 算 例 题

例 1，一实心变厚旋转圆盘，其厚度 $t=cr^{-1}$，$v=0.3$，划分单元数 $N=10$，半径为 a，计算出各点的 u、σ_r 和 σ_θ 如表 1 所示。

表 1

r/a	$u\times\frac{\omega^2\rho_1}{E}a^3$		$\sigma_r\times\omega^2\rho_1a^2$		$\sigma_\theta\times\omega^2\rho_1a^2$	
	精确解	本文解	精确解	本文解	精确解	本文解
0.0	0	0.0	0	0.00538	0	0.00538
0.1	0.00058	0.00054	0.11929	0.12056	0.09005	0.090307
0.2	0.01729	0.01729	0.18360	0.18338	0.14153	0.14145
0.3	0.03299	0.03300	0.22314	0.22322	0.17693	0.17694
0.4	0.05075	0.05076	0.24244	0.24243	0.19962	0.19961
0.5	0.06901	0.06901	0.24341	0.24342	0.21104	0.21104

r/a	$u\times\frac{\omega^2\rho_1}{E}a^3$		$\sigma_r\times\omega^2\rho_1a^2$		$\sigma_\theta\times\omega^2\rho_1a^2$	
	精确解	本文解	精确解	本文解	精确解	本文解
0.6	0.08630	0.08631	0.22712	0.22712	0.21198	0.21198
0.7	0.10126	0.10126	0.19425	0.19425	0.20293	0.20293
0.8	0.11254	0.11254	0.14253	0.14253	0.18424	0.18424
0.9	0.11882	0.11882	0.08040	0.08040	0.15614	0.15614
1.0	0.11882	0.11881	0	0.0	0.11882	0.11882

例 2，一空心变厚旋转圆盘，其厚度 $t=cr^{-1.5}$，$\upsilon=0.3$，划分单元数 $N=10$，外半径为 a，内半径 $b=0.2a$，计算出各点的 u、σ_r 和 σ_θ 如表 2 所示。

表　2

r/a	$u\times\frac{\omega^2\rho_1}{E}a^3$		$\sigma_r\times\omega^2\rho_1a^2$		$\sigma_\theta\times\omega^2\rho_1a^2$	
	精确解	本文解	精确解	本文解	精确解	本文解
0.20	0.07061	0.07056	0	0	0.35280	0.35305
0.28	0.06721	0.06737	0.09123	0.09190	0.26741	0.26819
0.36	0.07053	0.07093	0.14399	0.14405	0.23912	0.24023
0.44	0.07803	0.07819	0.17660	0.17530	0.23052	0.23029
0.52	0.08747	0.08745	0.19022	0.19025	0.22527	0.22524
0.60	0.09746	0.09744	0.19102	0.19105	0.21974	0.21972
0.68	0.10715	0.10713	0.17848	0.17850	0.21112	0.21110
0.76	0.11557	0.11556	0.15299	0.15300	0.19796	0.19795
0.84	0.12182	0.12181	0.11470	0.11471	0.17943	0.17943
0.92	0.12502	0.12501	0.06370	0.06371	0.15500	0.15499
1.0	0.12427	0.12428	0	0	0.12427	0.12428

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DERIVATION OF DIMNESIONLESS EQUATIONS FROM METHOD OF ANALYSIS OF DIRECTIONAL UNIT SYSTEM

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ABSTRACT In this paper the dimensionless equations are derived from the method of analysis of directional unit system ,and two examples are presented in illustration . The important priciple is suggested that the choice and treatment of the unit system in the similarity theory is quite differ from those of public unit systems such as SI system or engineering unit system, even though all they must conform to the Gauss' s rule of selection of unit system. For the former the flexibility and the particularity is necessary, because in the similarity theory the particular and actually phenomenon is investigated, and for the latter the invariance and the universality is necessary.as it is based on possibility of occurrance of all phenomena.

APPLICATION OF FINITE-ELEMENT METHOD TO CALCULATING STRESS FIELD IN NOZZLE INSERT OF SOLID ROCKET MOTOR

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ABSTRACT The finite-element method is applied to analyzing the two-dimensional stress field in the graphite insert of a composite nozzle under pressure and transient temperature loads in a solid rocket motor. The mechanical models, basic approaches and numerical examples are presented in this paper. The anisotropy of the material is considered in the computation. A compression technique of variable bandwidth is applied to the stiffness matrix [K] of the integral structure in order to save main storage. The calculated results are agreed with the experimental data very well.

TECHNICAL NOTES

A SPLINE INTEGRAL EQUATION METHOD FOR ROTATING VARIABLE-THICKNESS DISC

Liu Xingye and Zheng Jianjun

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ABSTRACT The integral equations of rotating disc with variable thickness are derived by means of the ring fundamental solution of elacticity. Then the integral equations are solved by spline function method. Also, this method is applied to the rotating variable-thickness disc with circular

hole. Its computing program is so simple as to work on the IBM-PC computer. Due to the spline interpolation, very high accuracy of the numerical solution can be achieved even if few elements are chosen. The numerical results obtained by this method is almost completely consistent with the exact solution. Obviously, this method overcomes the shortcomings of the finite element method or finte difference method in need of solving huge systems of algebraical equations.

OPTIMIZATION OF DISPERSED ELEMENT NODE SERIAL NUMBER

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ABSTRACT The computer time and storage necessary for structure finite-element analysis are mainly dependent on the bandwidth of the general rigidity matrix [K], if its dimension is fixed. According to the sparse characteristic of the general rigidity matrix and the principle of decreasing matrix bandwidth, an optimization of element node serial number is provided. It is simple and suitable for different elements, not only for the single-loop, but also for the multi-loop.

AN INVESTIGATION ON STAGNATION PRESSURE ERRORS DUE TO ROTATION STATE BEHIND A ROTOR

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ABSTRACT This paper proposes that the errors in stagnation pressure measurement due to rotation state behind a rotor should include perception errors in addition to the weighting error and dynamic response errors. The analysis and experimental data show that the stagnation pressure in wake may be higher than that in the main flow, so the weighting error may be positive. These two conclusions are contrary to the affimation of reference [1]. This paper also puts forward various approaches to minimize the errors due to the rotation state.

ACTIVE CONTROL OF INLET DISTORTED FLOW FIELD IN COMPRESSOR INLET

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ABSTRACT The feasibility of alleviating the destabilizing effect of the inlet circumferential total pressure distortion on an axial compressor by active control of individually adjustable inlet guide vanes (IAIGV) has been qualitatively analyzed under the parallel compressor hypothesis. The result shows that when the distortion is quasi-steady it is possible to use the actively controlled IAIGV with a relatively simple proportional control scheme to adjust the flow angle, and thus to prevent the compressor entering the unstable state. For a typical blade setting the angular adjustment of IAIGV needed to cope with 10% total pressure distortion is of only few degrees.